Applications of Geometric Algebra I

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3D Algebra

- 3D basis consists of 8 elements
- Represent lines, planes and volumes, from a common origin

Grade 0	Grade 1	Grade 2	Grade 3
Scalar	Vector	Bivector	Trivector
1	e_1, e_2, e_3	e_1e_2, e_2e_3, e_3e_1	I

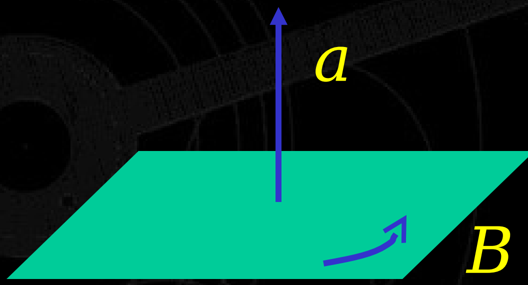
Algebraic Relations

- Generators anticommute $e_1 e_2 = -e_2 e_1$
- Geometric product $ab = a \cdot b + a \wedge b$
- Inner product $a \cdot b = \frac{1}{2}(ab + ba)$
- Outer product $a \wedge b = \frac{1}{2}(ab - ba)$
- Bivector norm $(e_1 \wedge e_2)^2 = -1$
- Trivector $I = e_1 e_2 e_3$
- Trivector norm $I^2 = -1$
- Trivectors commute with all other elements

Lines and Planes

- Pseudoscalar gives a map between lines and planes

$$B = Ia$$
$$a = -IB$$



- Allows us to recover the vector (cross) product

$$a \times b = -I a \wedge b$$

- But lines and planes are different
- Far better to keep them as distinct entities

Quaternions

- For the bivectors set

$$i = e_2e_3, \qquad j = -e_3e_1, \qquad k = e_1e_2$$

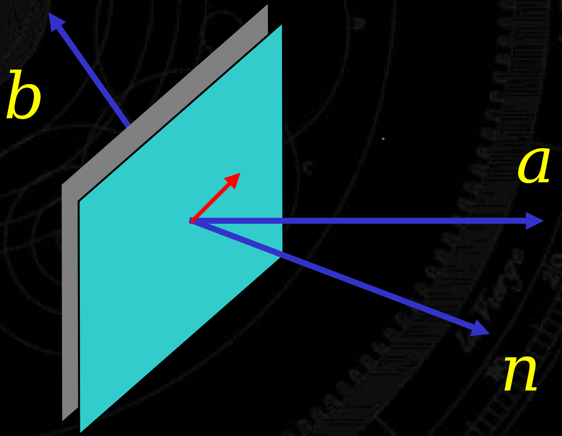
- These satisfy the quaternion relations

$$i^2 = j^2 = k^2 = ijk = -1$$

- So quaternions embedded in 3D GA
- Do not lose anything, but
  - Vectors and planes now separated
  - Note the minus sign!
  - GA generalises

Reflections

- Build rotations from reflections
- Good example of geometric product – arises in ***operations***



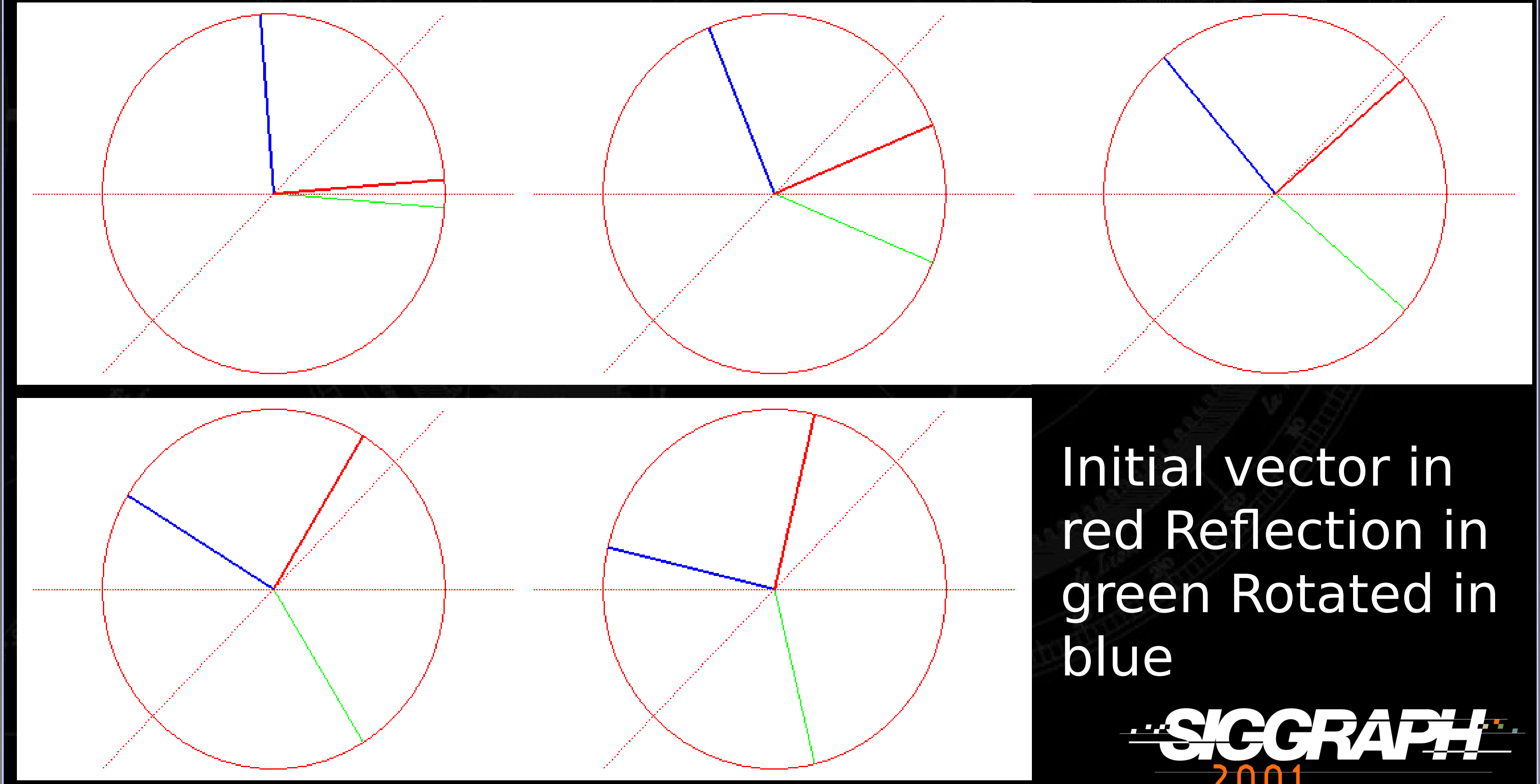
$$a_{\parallel} = (a \cdot n)n$$
$$a_{\perp} = a - (a \cdot n)n$$

- Image of reflection is

$$b = a_{\perp} - a_{\parallel} = a - 2(a \cdot n)n$$
$$= a - (an + na)n = \boxed{-nan}$$

Rotations

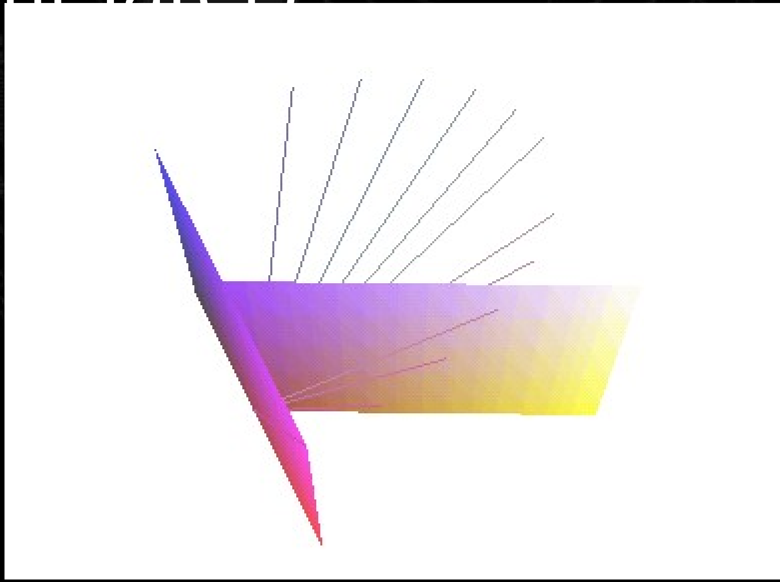
- 2 successive reflections give a rotation



Initial vector in red Reflection in green Rotated in blue

Rotations

- Direction perpendicular to the two reflection vectors is unchanged
- So far, will only talk about rotations in a plane with a fixed origin (more general treatment later)



Algebraic Formulation

- Now look at the algebraic expression for a pair of reflections

$$a \mapsto -m(-nan)m = mnanm$$

- Define the **rotor** $R = mn$
- Rotation encoded algebraically by

$$\boxed{a \mapsto RaR^\dagger} \qquad R^\dagger = nm$$

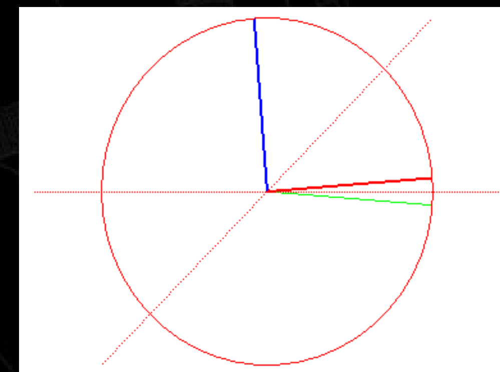
- Dagger symbol used for the **reverse**

Rotors

- Rotor is a geometric product of 2 unit vectors $R = mn = \cos\theta + m \wedge n$
- Bivector has square $(m \wedge n)^2 = (mn - \cos\theta)(\cos\theta - nm) = -\sin^2\theta$
- Used to the negative square by now! $B = \dfrac{m \wedge n}{\sin\theta}$
- Introduce unit bivector
- Rotor now written $R = \cos\theta + \sin\theta\, B$

Exponential Form

- Can now write $R = \exp(\theta \hat{B})$
- But:
  - rotation was through **twice** angle between the vectors
  - Rotation went with orientation $n \to m$



- Correct these, get double-sided, half-angle formula

$$a' = R a \tilde{R} \qquad R = \exp(-\theta \hat{B}/2)$$

- Completely general!

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Rotors in 3D

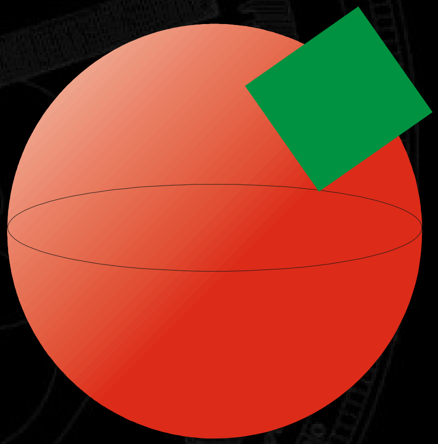
- Can rewrite in terms of an axis via

$$R = \exp(-\theta I n/2)$$

- Rotors even grade (scalar + bivector in 3D)
- Normalised: $R\tilde{R} = mnnm = 1$
- Reduces d.o.f. from 4 to 3 – enough for a rotation
- In 3D a rotor is a normalised, even element
- The same as a unit quaternion

Group Manifold

- Rotors are elements of a 4D space, normalised to 1
- They lie on a **3-sphere**
- This is the **group manifold**
- **Tangent space** is 3D
- Natural **linear** structure for rotors
- Rotors R and $-R$ define the same rotation
- Rotation group manifold is more complicated



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Comparison

- Euler angles give a standard parameterisation of rotations

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

- Rotor form far easier

$$R = \exp(-e_1e_2\phi/2)\exp(-e_2e_3\theta/2)\exp(-e_1e_2\psi/2)$$

- But can do better than this anyway – work directly with the rotor element

Composition

- Form the compound rotation from a pair of successive rotations

$$a \mapsto R_2 R_1 a \tilde{R}_1 \tilde{R}_2$$

- Compound rotor given by group combination law $R = R_2 R_1$
- Far more efficient than multiplying matrices
- More robust to numerical error
- In many applications can safely ignore the normalisation until the final step

Oriented Rotations

- Rotate through 2 different orientations
- Positive Orientation

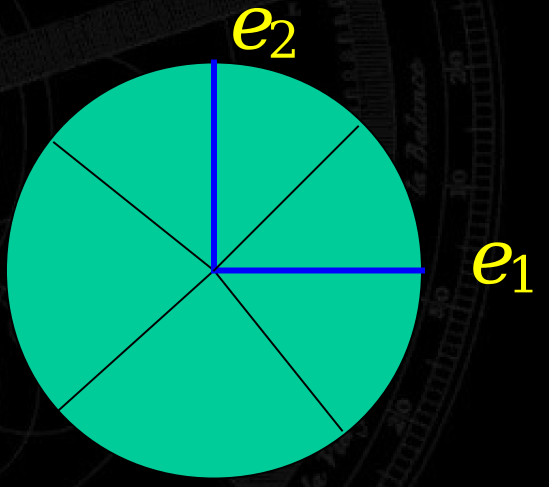
$$R = \exp(-\theta e_1 e_2/2)$$

$$= \exp(-e_1 e_2 \pi/4)$$

- Negative Orientation

$$S = \exp(\theta e_1 e_2/2)$$

$$= \exp(e_1 e_2 3\pi/4) = -R$$



- So R and $-R$ encode the same absolute rotation, but with different orientations

Lie Groups

- Every rotor can be written as $\exp(-B/2)$
- Rotors form a continuous (Lie) group
- Bivectors form a **Lie algebra** under the commutator product
- **All** finite Lie groups are rotor groups
- **All** finite Lie algebras are bivector algebras
- (Infinite case not fully clear, yet)
- In conformal case (later) starting point of screw theory (Clifford, 1870s)!

Interpolation

- How do we interpolate between 2 rotations?
- Form path between rotors

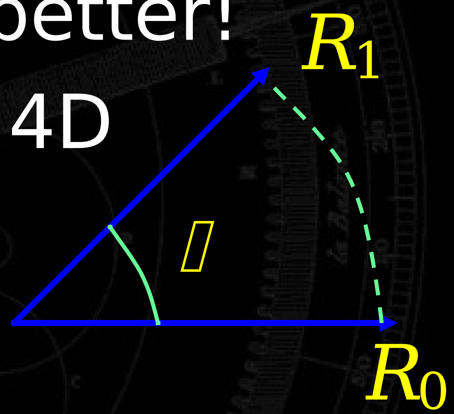
$$R(0) = R_0$$
$$R(1) = R_1$$

$$R(\lambda) = R_0 \exp(\lambda B)$$

- Find B from $\exp(B) = \tilde{R}_0 R_1$
- This path is invariant. If points transformed, path transforms the same way
- Midpoint simply $R(1/2) = R_0 \exp(B/2)$
- Works for ***all*** Lie groups

Interpolation - SLERP

- For rotors in 3D can do even better!
- View rotors as unit vectors in 4D
- Path is a circle in a plane
- Use simple trig’ to get SLERP



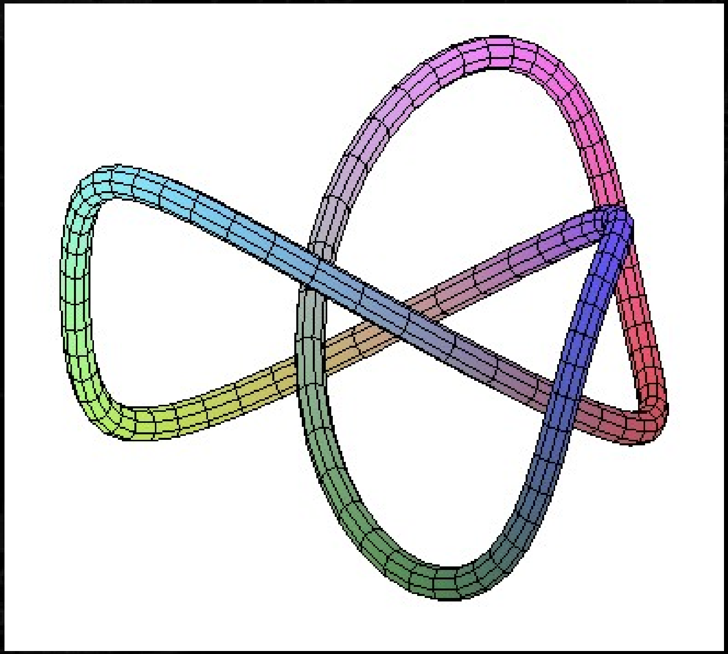
$$R(\lambda) = \frac{1}{\sin(\theta)}\left(\sin((1-\lambda)\theta)R_0 + \sin(\lambda\theta)R_1\right)$$

- For midpoint add the rotors and normalise!

$$R(1/2) = \frac{\sin(\theta/2)}{\sin(\theta)}\boxed{(R_0 + R_1)}$$

Applications

- Use SLERP with spline constructions for general interpolation
- Interpolate between series of rigid-body orientations
- Elasticity
- Framing a curve
- Extend to general transformations



Linearisation

- Common theme is that rotors can **linearise** the rotation group, without approximating!
- Relax the norm constraint on the rotor and write $RA\tilde{R} = \psi A \psi^{-1}$
- ψ belongs to a linear space. Has a natural **calculus**.
- Very powerful in optimisation problems involving rotations
- Employed in computer vision algorithms

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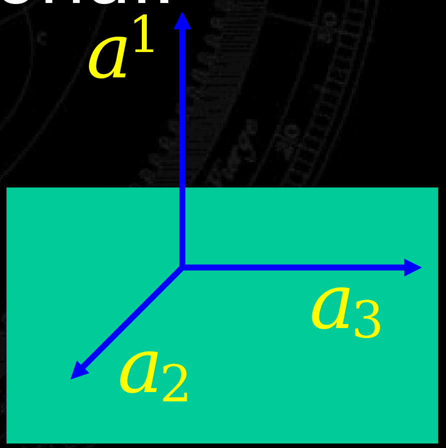
Recovering a Rotor

- Given two sets of vectors related by a rotation, how do we recover the rotor?
- Suppose $b_i = R a_i \tilde{R}$
- In general, assume not orthogonal.
- Need reciprocal frame



$$a^1 = \frac{a_2 \wedge a_3 I}{(a_1 \wedge a_2 \wedge a_3 I)}$$

- Satisfies $a^i \cdot a_j = \delta^i_j$

Recovering a Rotor II

- Now form even-grade object

  $$b_i a^i = R a_i (\alpha - B) a^i = R(3\alpha + B) = -1 + 4\alpha R$$

- Define un-normalised rotor

  $$\psi = b_i a^i + 1$$

- Recover the rotor immediately now as

  $$R = \frac{\psi}{|\psi|}$$

- Very efficient, but
  - May have to check the sign
  - Careful with 180° rotations

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Rotor Equations

- Suppose we take a path in rotor space $R(\lambda)$
- Differentiating the constraint tells us that

$$\frac{d}{d\lambda}(R\tilde{R}) = R'\tilde{R} + R\tilde{R}' = 0$$

- Re-arranging, see that

$$R'\tilde{R} = -(R'\tilde{R})^{\sim} = \text{Bivector}$$

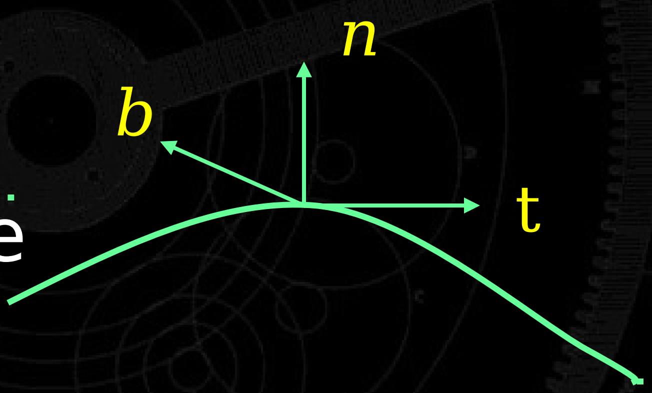
- Arrive at **rotor equation**

$$\boxed{R' = -\frac{1}{2}BR}$$

- This is totally general. Underlies the theory of **Lie groups**

Example

- As an example, return to framing a curve.
- Define Frenet frame
- Relate to fixed frame $(t, n, b) = R e_i \tilde{R}$
- Rotor equation $R' = \frac{1}{2} R \Omega, \quad \Omega = \kappa_1 e_2 e_1 + \kappa_2 e_3 e_2$
- Rotor equation in terms of curvature and torsion



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Linearisation II

- Rotor equations can be awkward (due to manifold structure)
- Linearisation idea works again
- Replace rotor with general element and write

$$\psi' = -\frac{1}{2}B\psi$$

- Standard ODE tools can now be applied (Runge-Kutta, etc.)
- Normalisation of ψ gives useful check on errors

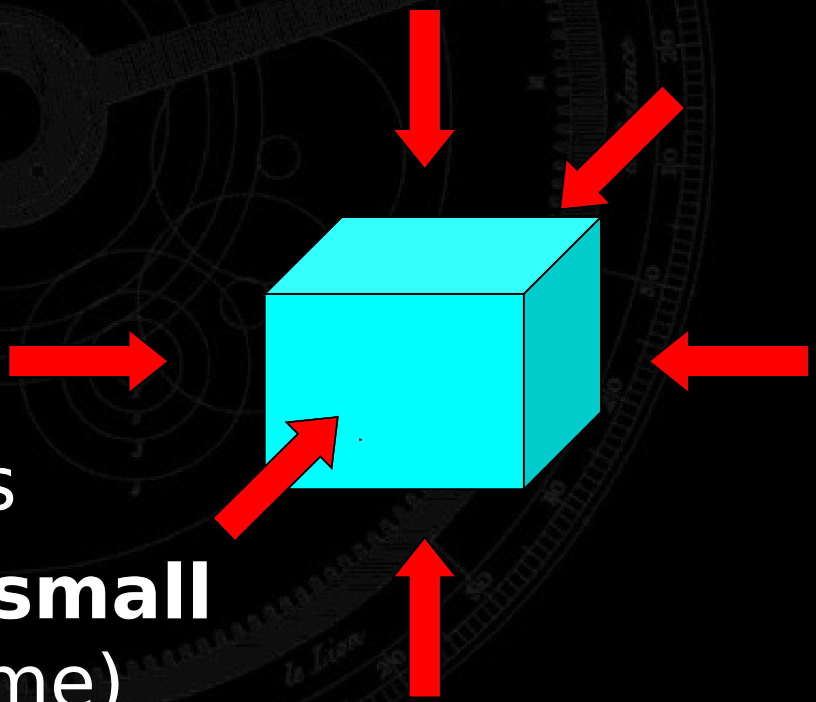
Elasticity

- Some basics of elasticity (solid mechanics):
  - When an object is placed under a **stress** (by stretching or through pressure) it responds by changing its shape.
  - This creates **strains** in the body.
  - In the linear theory stress and strain are related by the **elastic constants**.
  - An example is Hooke's law $F=-kx$, where k is the spring constant.
  - Just the beginning!

Bulk Modulus

- Place an object under uniform pressure *P*
- Volume changes by

$$-\Delta P = B\frac{\Delta V}{V}$$

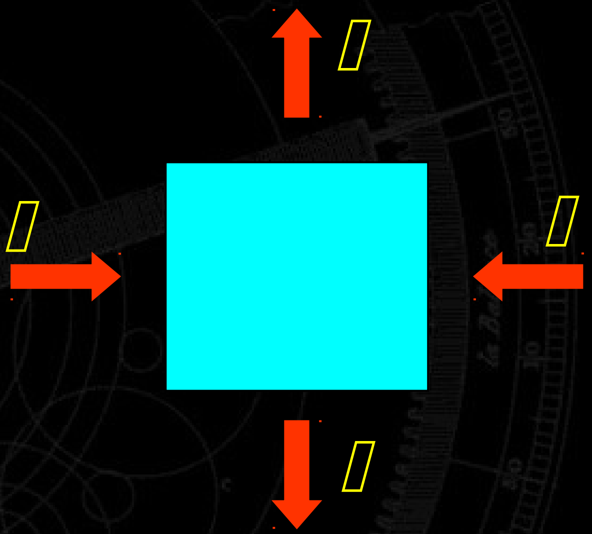


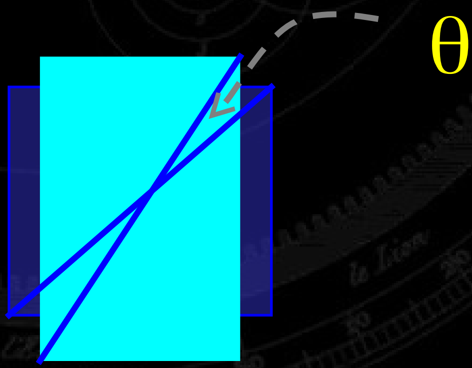
- *B* is the bulk modulus
- Definition applies for **small** pressures (linear regime)

Shear Modulus

- Sheers produced by combination of tension and compression
- Sheer modulus G is Shear stress / angle

$$G \;\square\; \frac{\square}{2\square}$$



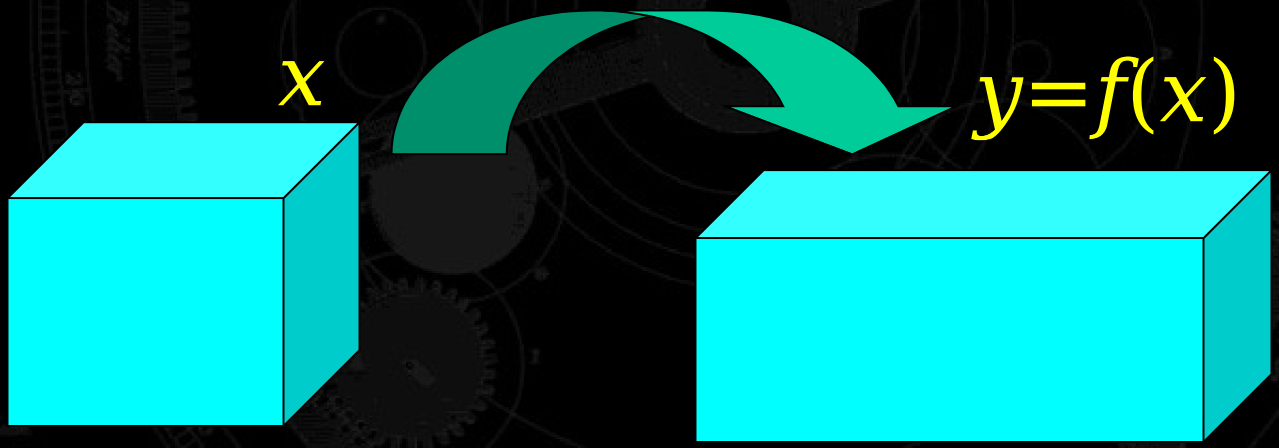


LIH Media

- The simplest elastic systems to consider are **linear, isotropic** and **homogeneous** media.
- For these, B and G contain all the relevant information.
- There are many ways to extend this:
  - Go beyond the linearised theory and treat large deflections
  - Find simplified models for rods and shells

Foundations

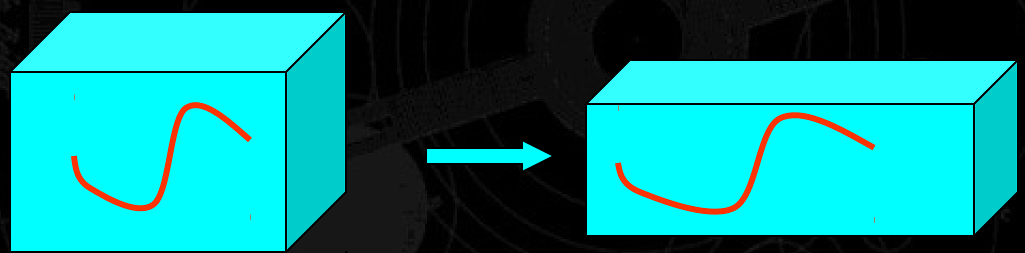
- Key idea is to relate the spatial configuration to a 'reference' copy.



- $y=f(x)$ is the **displacement** field. In general, this will be time-dependent as well.

Paths

- From *f(x)* we want to extract information about the strains. Consider a path



- Tangent vectors map to

$$f(x+\varepsilon a)-f(x)\approx\varepsilon a\cdot\nabla f(x)=\varepsilon F(a)$$

- $F(a)=F(a;x)$ is a linear function of a. Tells us about local distortions.

Path Lengths

- Path length in the reference body is

$$\int \left(\frac{dx}{d\lambda} \cdot \frac{dx}{d\lambda} \right)^{1/2} d\lambda$$

- This transforms to

$$\int (\mathsf{F}(x') \cdot \mathsf{F}(x'))^{1/2} d\lambda$$

- Define the function $\mathsf{G}(a)$, acting entirely in the reference body, by

$$\mathsf{G}(a) = \overline{\mathsf{F}}\mathsf{F}(a)$$

The Strain Tensor

- For elasticity, usually best to ‘pull’ everything back to the reference copy
- Use same idea for rigid body mechanics
- Define the strain tensor from $G(a)$
  - Most natural is

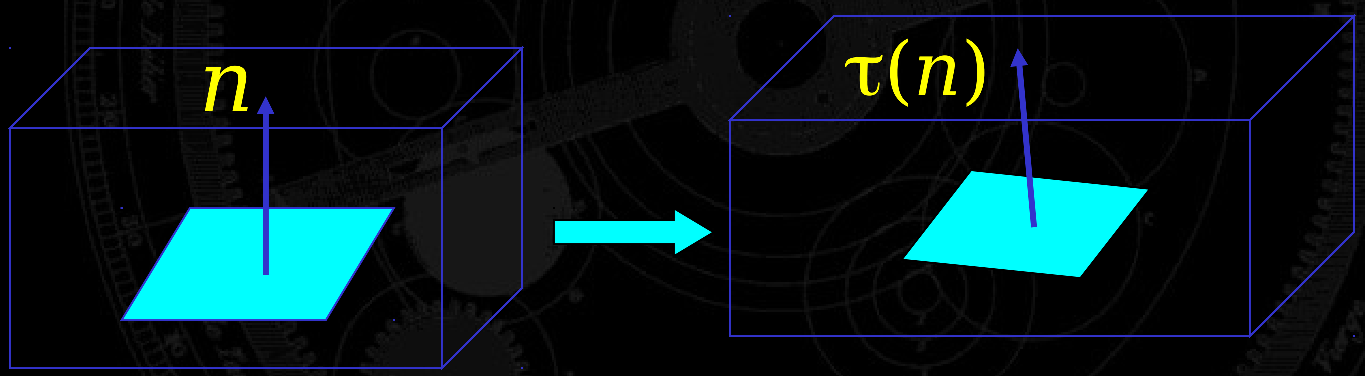
$$E(a) = \frac{1}{2}\left(G(a) - a\right)$$

  - An alternative (rarely seen) is

$$E(a) = \frac{1}{2}\ln G(a)$$

The Stress Tensor

- Contact force between 2 surfaces is a linear function of the normal (Cauchy)



- $\tau(n)=\tau(n;x)$ returns a vector in the material body. 'Pull back' to reference copy to define $T(n) = F^{-1}\tau(Fn)$

Constitutive Relations

- Relate the stress and the strain tensors in the reference configuration
- Considerable freedom in the choice here
- The simplest, LIH media have

$$\mathrm{T}(a) = 2G\mathrm{E}(a) + \left(B - \tfrac{2}{3}G\right)(\mathbf{tr}\,\mathrm{E})\,a$$

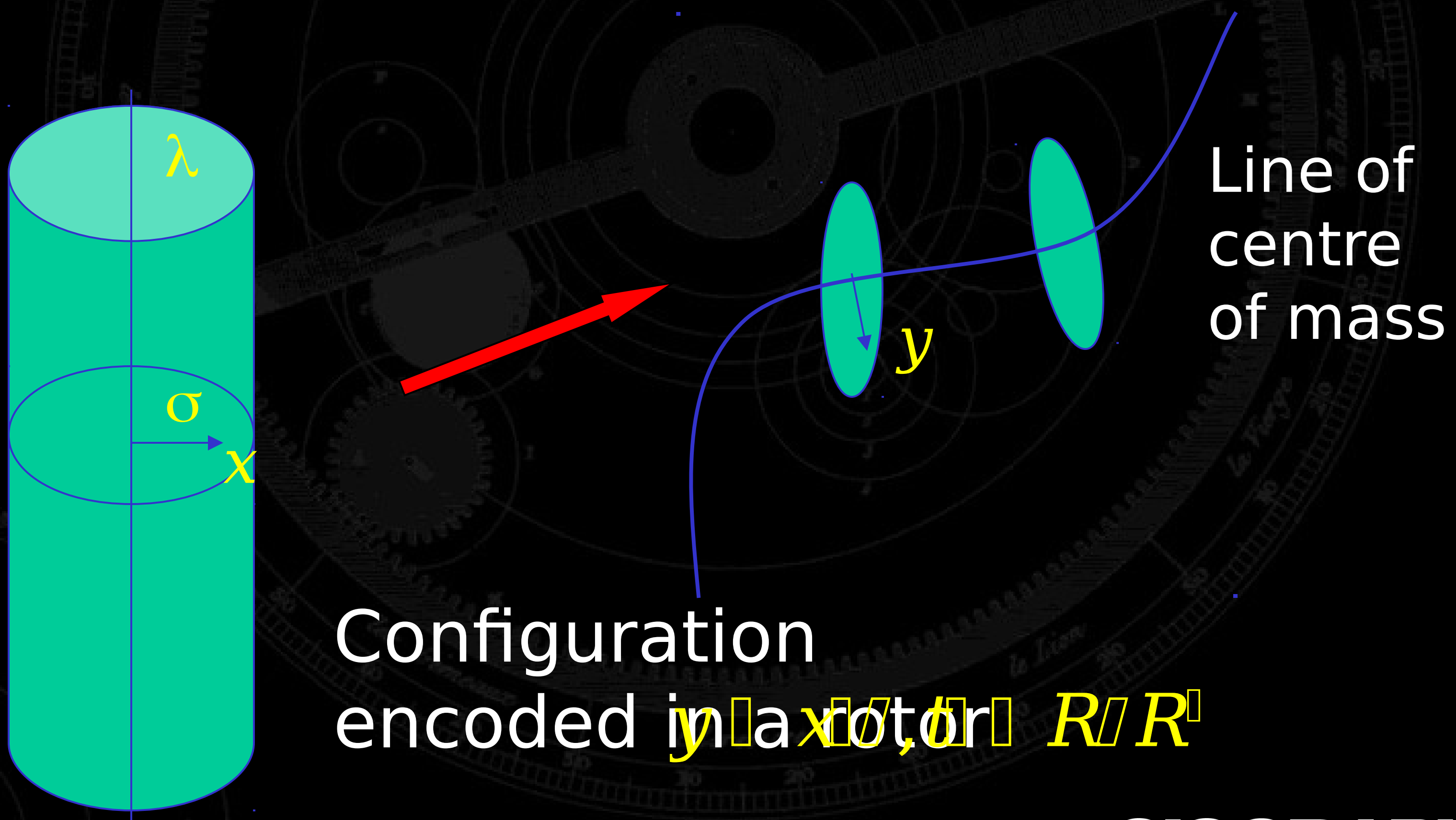
- Can build up into large deflections
- Combined with balance equations, get full set of dynamical equations
- Can get equations from an action principle

Problems

- Complicated, and difficult numerically
- In need of some powerful advanced mathematics for the full nonlinear theory (FEM...)
- Geometric algebra helps because it
  - is coordinate free
  - integrates linear algebra and calculus smoothly
- But need simpler models
- Look at models for rods and beams

Deformable Rod

- Reference configuration is a cylinder



Configuration encoded in a rotor $y(x,\sigma,t) = R x \tilde{R}$

Technical Part

- Spare details, but:
- Write down an action integral
- Integrate out the coordinates over each disk
- Get (variable) **bending moments** along the centre line
- Carry out variational principle
- Get set of equations for the rotor field
- Can apply to static or dynamic configurations

Simplest Equations

- Static configuration, and ignore stretching
- Have rotor equation

$$\frac{dR}{d\lambda} = -\frac{1}{2} R \Omega_B$$

- Find bivector from applied couple and elastic constants. $\mathsf{I}(B)$ is a known linear function of these mapping bivectors to bivectors

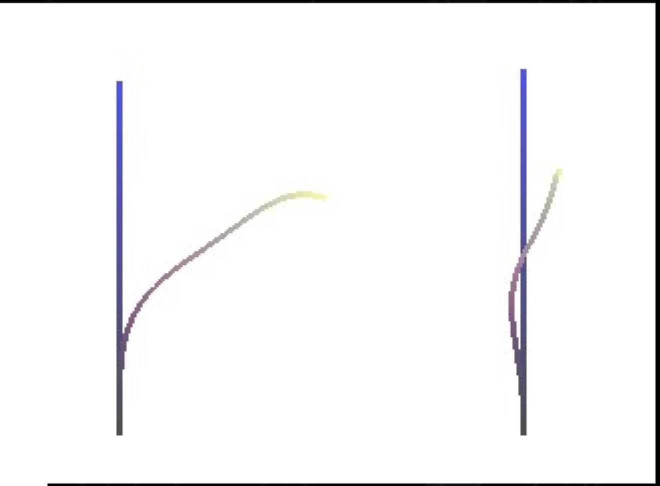
$$\Omega_B = \mathsf{I}^{-1}(\tilde{R} C R)$$

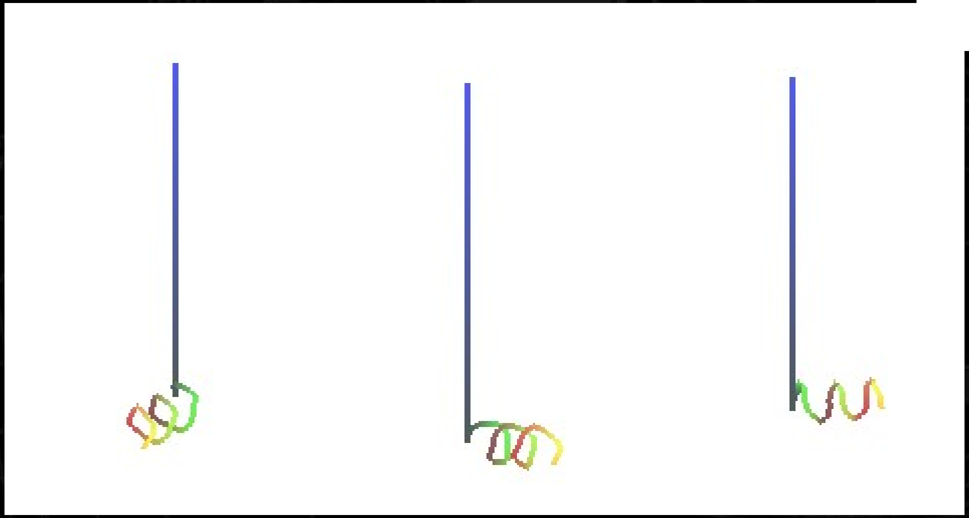
- Integrate to recover curve

$$x' = R e_1 \tilde{R}$$

Example

- Even this simple set of equations can give highly complex configurations!





Small, linear deflections build up to give large deformations

Summary

- Rotors are a general purpose tool for handling rotations in arbitrary dimensions
- Computationally more efficient than matrices
- Can be associated with a linear space
- Easy to interpolate
- Have a natural associated calculus
- Form basis for algorithms in elasticity and computer vision
- All this extends to general groups!

Further Information

- All papers on Cambridge GA group website: www.mrao.cam.ac.uk/~clifford
- Applications of GA to computer science and engineering are discussed in the proceedings of the AGACSE 2001 conference. www.mrao.cam.ac.uk/agacse2001
- IMA Conference in Cambridge, 9th Sept 2002
- ‘Geometric Algebra for Physicists’ (Doran + Lasenby). Published by CUP, soon.

Revised Timetable

- 8.30 – 9.15 Rockwood *Introduction and outline of geometric algebra*
- 9.15 – 10.00 Mann *Illustrating the algebra I*
- 10.00 -10.15 Break
- 10.15 – 11.15 Doran *Applications I*
- 11.15 – 12.00 Lasenby *Applications II*
- 1.30 – 2.00 Doran *Beyond Euclidean Geometry*
- 2.00 – 3.00 Hestenes *Computational Geometry*
- 3.00 – 3.15 Break
- 3.15 – 4.00 Dorst *Illustrating the algebra II*
- 4.00 – 4.30 Lasenby *Applications III*
- 4.30 Panel